Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 260 (2011) 1925–1943
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

On the double commutant of Cowen–Douglas operators

Li Chen [a,*], Ronald G. Douglas [b,1], Kunyu Guo [c,2]

[a] *Department of Mathematics, Tianjin University, Tianjin 300072, China*
[b] *Department of Mathematics, Texas A&M University, College Station, TX 77843, United States*
[c] *School of Mathematics Science, Fudan University, Shanghai 200433, China*

Received 27 December 2009; accepted 28 December 2010

Available online 8 January 2011

Communicated by Gilles Godefroy

Abstract

Let T be a Cowen–Douglas operator. In this paper, we study the von Neumann algebra $V^*(T)$ consisting of operators commuting with both T and T^* from a geometric viewpoint. We identify operators in $V^*(T)$ with connection-preserving bundle maps on $E(T)$, the holomorphic Hermitian vector bundle associated to T. By studying such bundle maps, the structure of $V^*(T)$ as well as information on reducing subspaces of T can be determined.

Keywords: Cowen–Douglas operator; Von Neumann algebra; Reducing subspace; Connection

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. Given a domain (connected open subset) Ω in $\mathbb{C}$ and a positive integer n, M.J. Cowen and the second author [2] introduced the operator class $\mathcal{B}_n(\Omega)$, consisting of operators T on $\mathcal{H}$ satisfying:

* Corresponding author.
E-mail addresses: lichen1019@gmail.com (L. Chen), rdouglas@math.tamu.edu (R.G. Douglas), kyguo@fudan.edu.cn (K. Guo).
1 Research supported in part by a grant from the National Science Foundation (US).
2 This work is partially supported by Laboratory of Mathematics for Nonlinear Science, Fudan University, and NSFC (10525106), NKBRPC (2006CB805905).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.12.030

(i) $\Omega \subseteq \sigma(T)$;
(ii) $\operatorname{ran}(T-w)=\mathcal{H}$ for w in Ω;
(iii) $\bigvee_{w\in\Omega}\ker(T-w)=\mathcal{H}$;
(iv) $\dim\ker(T-w)=n$ for w in Ω.

Given an operator T in $\mathcal{B}_n(\Omega)$, the mapping $w\to\ker(T-w)$ defines a rank n holomorphic Hermitian vector bundle over Ω, which we denote by $E(T)$. An important observation in [2] is that invariants of T can be revealed by investigating their geometric counterparts in $E(T)$.

Our main aim in this paper is to study the von Neumann algebra $V^*(T)$ of operators commuting with both T and T^* for T in $\mathcal{B}_n(\Omega)$. There are several motivations for our investigation.

For one thing, $\mathcal{B}_n(\Omega)$ contains many important classes of operators and characterizing reducing subspaces of these operators is an interesting topic in operator theory. Our investigation arises from the study of multiplication operators on Hilbert spaces consisting of holomorphic functions. For a typical example we mention the multiplication operator M_B on the Bergman space where B is a finite Blaschke product. In this case, the adjoint of M_ϕ is a Cowen–Douglas operator. An open conjecture is that if B is a Blaschke product of order n, then M_B has at most n distinct minimal reducing subspaces, or in language of operator algebra, the von Neumann algebra $V^*(M_B)$ has at most n minimal projections. The algebra $V^*(M_B)$ is finite dimensional (see [4]), and using general theory of finite dimensional von Neumann algebras, one can show that the conjecture is equivalent to the statement that $V^*(M_B)$ is abelian (see [4,7] for detailed discussions). Progress along this line can also be found in [9,10,14]. For further discussion on the relation between operator theory on function spaces and von Neumann algebras, see [6] and [8].

We will not go any further on concrete problems, which however, suggest that it is worthwhile to have a conceptual understanding of $V^*(T)$ for an arbitrary Cowen–Douglas operator T.

Another reason for studying $V^*(T)$ lies in its close relation to the differential geometry of the bundle $E(T)$. Recall that if S is an operator commuting with T, then $S\ker(T-w)\subseteq\ker(T-w)$, and hence S induces a holomorphic bundle map on $E(T)$ which we denote by $\Gamma(S)$. If S lies in $V^*(T)$, then $\Gamma(S)$ is not only holomorphic, but also connection-preserving, as we shall see later.

Projections in $V^*(T)$, or reducing subspaces of T, are in one-to-one correspondence with *reducing subbundles* of $E(T)$. (We say a subbundle F of a holomorphic Hermitian vector bundle E is a reducing subbundle if both F and its orthogonal complement $F^\perp$ in E are *holomorphic* subbundles.) Now we briefly describe this correspondence (see [2] for details):

If $\mathcal{H}_1$ is a reducing subspace for T in $\mathcal{B}_n(\Omega)$ and $\mathcal{H}_2=\mathcal{H}_1^\perp$, then $T|_{\mathcal{H}_1}$ and $T|_{\mathcal{H}_2}$ are both Cowen–Douglas operators. In this case, $E(T|_{\mathcal{H}_1})$ and $E(T|_{\mathcal{H}_2})$ are mutually orthogonal holomorphic subbundles such that

$$E(T)=E(T|_{\mathcal{H}_1})\oplus E(T|_{\mathcal{H}_2}).$$

Conversely, if $E(T)$ can be decomposed into an orthogonal direct sum of two holomorphic subbundles E_1 and E_2, then there exist reducing subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ with $\mathcal{H}_1=\bigvee_{w\in\Omega}E_{1w}$ and $\mathcal{H}_2=\bigvee_{w\in\Omega}E_{2w}$, where E_{iw} denotes the fibre of E_i at w.

Two reducing subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ for T are said to be unitarily equivalent if there exists a unitary operator $U:\mathcal{H}_1\to\mathcal{H}_2$ such that $UT|_{\mathcal{H}_1}=T|_{\mathcal{H}_2}U$. A key result in [2], which we restate in the following, asserts that $\mathcal{H}_1$ and $\mathcal{H}_2$ are unitarily equivalent if and only if there exists an isomorphic holomorphic bundle map between $E(T|_{\mathcal{H}_1})$ and $E(T|_{\mathcal{H}_2})$.

Theorem 1.1. *(See [2].) Two Cowen–Douglas operators T_1 and T_2 in $\mathcal{B}_n(\Omega)$ are unitarily equivalent if and only if there exists a local isometric holomorphic bundle map Φ from $E(T_1)$ to $E(T_2)$. In this case, $\Phi = \Gamma(U)$ where U is the intertwining unitary operator.*

Remark 1.2. We say that two holomorphic Hermitian vector bundles over Ω are locally equivalent if there exists an isometric holomorphic bundle map Φ defined on an open subset Δ in Ω between them. The theorem says that the bundle map Φ defined on Δ can be extended to a globally defined map $\Gamma(U)$; in other words, local equivalence implies global equivalence. This arises from the uniqueness of analytic continuation and the well-known spanning property (see [2] for a proof) that

$$\bigvee_{w \in \Delta} \ker(T - w) = \mathcal{H},$$

for any open subset Δ in Ω.

Given a Cowen–Douglas operator T, Theorem 1.1 asserts that holomorphic isometric bundle maps on $E(T)$ are in one-to-one correspondence with unitary operators in $V^*(T)$. In Section 3, we generalize this correspondence to connection-preserving bundle maps on $E(T)$ and $V^*(T)$ (holomorphic isometric bundle maps are necessarily connection-preserving, as we shall see in the next section). Our result is stated as follows:

Theorem 1.3. *Let T be a Cowen–Douglas operator in $\mathcal{B}_n(\Omega)$ and Φ be a bundle map on $E(T)$. There exists an operator T_Φ in $V^*(T)$ such that $\Phi = \Gamma(T_\Phi)$ if and only if Φ is connection-preserving. Consequently, the map Γ is a $*$-isomorphism from $V^*(T)$ to connection-preserving bundle maps on $E(T)$.*

In Section 4, by studying connection-preserving bundle maps on $E(T)$, we show that $V^*(T)$ is isomorphic to the commutant of a matrix algebra. This matrix algebra represents the algebra of bundle maps on $E(T)$ generated by curvature and its covariant derivatives to all orders. Our discussions are based on a result called "block diagonalization of connections" established by Cowen and the second author [3] where they studied the equivalence problem of C^∞ Hermitian vector bundles. We will also use this result to study reducing subbundles of $E(T)$, which provides a canonical decomposition of $\mathcal{H}$ into the direct sum of minimal reducing subspaces. As a complementary example, we discuss a typical kind of Cowen–Douglas operators, called the bundle shifts, which represent a large class of subnormal operators related to multiply-connected domains [1].

2. Preliminaries on Hermitian vector bundles

In this section, we provide necessary preliminaries on Hermitian vector bundles, which are mainly extracted from [2]. General references can be found in [11,13].

Given a domain Ω in $\mathbb{C}$, a rank n holomorphic vector bundle over Ω is a complex manifold E with a holomorphic map π from E onto Ω such that each fibre $E_\lambda = \pi^{-1}(\lambda)$ is a copy of $\mathbb{C}^n$ and for each λ_0 in Ω, there exists a neighborhood Δ of λ_0 and holomorphic functions $s_1, \dots, s_n$ from Δ to E such that $E_\lambda = \bigvee\{s_1(\lambda), \dots, s_n(\lambda)\}$. The n-tuple of functions $\{s_1, \dots, s_n\}$ is called a holomorphic frame over Δ. A section is a map s from an open subset of Ω to E such that $\pi(s(\lambda)) = \lambda$.

A bundle map between two bundles E_1 and E_2 defines a linear transformation from $E_{1\lambda}$ to $E_{2\lambda}$ for λ in Ω. Locally a bundle map can be represented by a matrix-valued function relative to the local frames of the two bundles. A bundle map between two holomorphic vector bundles is holomorphic if its representing matrix function relative to holomorphic frames is holomorphic. A holomorphic bundle map is determined by its restriction on any open subset Δ in Ω.

A Hermitian vector bundle is a vector bundle E such that each fibre E_λ is an inner product space. Given a bundle map Φ from a Hermitian vector bundle E_1 to E_2, we can define its adjoint to be a bundle map Φ^* from E_2 to E_1 satisfying

$$\langle \Phi s(\lambda), t(\lambda)\rangle_{E_{2\lambda}} = \langle s(\lambda), \Phi^* t(\lambda)\rangle_{E_{1\lambda}}$$

for any sections s and t of E_1 and E_2, respectively.

For a separable Hilbert space $\mathcal{H}$ and a positive integer n, let $\mathcal{G}r(n, \mathcal{H})$ denote the Grassmann manifold of all n-dimensional subspaces of $\mathcal{H}$. A map $f : \Omega \to \mathcal{G}r(n, \mathcal{H})$ is called a holomorphic curve if for any point λ_0 in Ω, there exists a neighborhood Δ of λ_0 and n holomorphic $\mathcal{H}$-valued functions $s_1, \dots, s_n$ on Δ such that $f(\lambda) = \bigvee\{s_1(\lambda), \dots, s_n(\lambda)\}$. A holomorphic curve naturally gives a Hermitian holomorphic vector bundle E_f over Ω. The fibre of E_f at a point λ is $f(\lambda)$ and the metric at each fibre is inherited from the inner product on $\mathcal{H}$. The local holomorphic functions $s_1, \dots, s_n$ form a holomorphic frame over Δ. It is shown in [2] that if T is a Cowen–Douglas operator in $\mathcal{B}_n(\Omega)$, the map $w \mapsto \ker(T - w)$ is a holomorphic curve and the resulting bundle is $E(T)$. In this paper, we concentrate on unitary invariants of holomorphic curves, while we would like to mention the work of Jiang and Ji [12], who studied the similarity questions rather than unitary ones and some of their methods are related to ours.

Let $\mathcal{E}(\Omega)$ denote the algebra of C^∞ functions on Ω and let $\mathcal{E}^p(\Omega)$ denote the C^∞ differential forms of degree p on Ω. Then we have $\mathcal{E}^0(\Omega) = \mathcal{E}(\Omega)$, $\mathcal{E}^1(\Omega) = \{f\,dz + g\,d\bar{z}\colon\ f, g \in \mathcal{E}(\Omega)\}$ and $\mathcal{E}^2(\Omega) = \{f\,dz\,d\bar{z}\colon\ f \in \mathcal{E}(\Omega)\}$.

For a C^∞ vector bundle E over Ω, let $\mathcal{E}^p(\Omega, E)$ denote the differential forms of degree p with coefficients in E, then $\mathcal{E}^0(\Omega, E)$ are just C^∞ sections of E on Ω.

A connection on E is a first order differential operator $D : \mathcal{E}^0(\Omega, E) \to \mathcal{E}^1(\Omega, E)$ such that

$$D(f\sigma) = df \otimes \sigma + f D(\sigma)$$

for f in $\mathcal{E}(\Omega)$ and σ in $\mathcal{E}^0(\Omega, E)$. The connection D is called *metric-preserving* if

$$d\langle \sigma_1, \sigma_2\rangle = \langle D\sigma_1, \sigma_2\rangle + \langle \sigma_1, D\sigma_2\rangle,$$

for σ_1, σ_2 in $\mathcal{E}^0(\Omega, E)$.

Locally, D can be represented by a connection matrix. Let $s = \{s_1, \dots, s_n\}$ be a local frame on Δ, then the connection matrix $\Theta(s) = [\Theta_{ij}]$ relative to the frame s is a matrix with 1-form entries Θ_{ij} defined on Δ such that

$$D(s_i) = \Sigma_{j=1}^n \Theta_{ij} \otimes s_j.$$

The connection D can be extended to a differential operator from $\mathcal{E}^1(\Omega, E)$ to $\mathcal{E}^2(\Omega, E)$ so that

$$D(\sigma \otimes \alpha) = D\sigma \wedge \alpha + \sigma \otimes d\alpha$$

for σ in $\mathcal{E}(\Omega, E)$ and α in $\mathcal{E}^1(\Omega)$.

It is well known that D^2 is C^∞ linear so we have for any σ in $\mathcal{E}(\Omega, E)$, that

$$D^2\sigma = \mathcal{K}\sigma\, dz\, d\bar{z},$$

where $\mathcal{K}$ is a bundle map on E which is uniquely determined by D^2. Thus D^2 can be identified with $\mathcal{K}$ and we call $\mathcal{K}$ the *curvature* of (E, D).

For a Hermitian vector bundle on a domain in $\mathbb{C}$, the curvature $\mathcal{K}$ is always *self-adjoint* provided that its defining connection D is metric-preserving (Section 2.15, [2]).

The matrix of D^2 relative to a frame s is given by

$$D^2(s) = d\Theta(s) + \Theta(s) \wedge \Theta(s). \tag{2.1}$$

Note that D is not a bundle map since it is not C^∞ linear, while it can be shown that the commutator of D with a bundle map is still a bundle map (Lemma 2.10, [2]). Thus for the bundle map Φ on E, there exists bundle maps Φ_z and $\Phi_{\bar{z}}$ satisfying

$$[D, \Phi] = D\Phi - \Phi D = \Phi_z \otimes dz + \Phi_{\bar{z}} \otimes d\bar{z}.$$

Then Φ_z and $\Phi_{\bar{z}}$ are called *covariant derivative* of Φ relative to the connection D. Since covariant derivatives are also bundle maps, we can continue this procedure to define higher order covariant derivatives $\Phi_{z^i\bar{z}^j}$ for all positive integers i, j.

The covariant derivatives of Φ and Φ^* are related as follows (Lemma 2.12, [2]):

$$(\Phi_z)^* = \left(\Phi^*\right)_{\bar{z}},\ (\Phi_{\bar{z}})^* = \left(\Phi^*\right)_z. \tag{2.2}$$

A bundle map Φ is called *connection preserving* if

$$[D, \Phi] = 0$$

or equivalently,

$$\Phi_z = \Phi_{\bar{z}} = 0.$$

By an easy computation (or see [3]), the matrix of $[D, \Phi]$ relative to a local frame s is $d\Phi(s) + [\Theta(s), \Phi(s)]$. Thus a bundle map is connection-preserving if and only if its matrix satisfies

$$d\Phi(s) + \left[\Theta(s), \Phi(s)\right] = 0. \tag{2.3}$$

An induction argument shows that a connection-preserving bundle map Φ necessarily preserves curvature as well as its covariant derivatives to all orders, i.e.

$$\Phi\mathcal{K}_{z^i\bar{z}^j} = \mathcal{K}_{z^i\bar{z}^j}\Phi$$

for all $0 \leqslant i, j < \infty$ (Remark 2.16, [2]).

Now we turn to holomorphic vector bundles. If E is a holomorphic Hermitian vector bundle, it is well known that there exists a unique canonical connection Θ on E, called the *Chern connection*, which is metric-preserving and compatible with the holomorphic structure.

Locally, given a holomorphic frame $s = \{s_1, \ldots, s_n\}$ with metric matrix $h = (\langle s_i, s_j \rangle)$,

$$\Theta(s) = \partial h h^{-1}. \tag{2.4}$$

The matrix of D^2 is given by

$$D^2(s) = \bar{\partial}\big(\partial h h^{-1}\big). \tag{2.5}$$

The matrix of the covariant derivatives of a bundle map Φ relative to this canonical connection is given by (sec 2.18, [2]):

$$\Phi_z(s) = \partial \Phi(s) + \big[\partial h h^{-1}, \Phi(s)\big] \quad \text{and} \quad \Phi_{\bar{z}}(s) = \bar{\partial}\Phi(s).$$

Thus the matrix of $\Phi_{\bar{z}}$ is just the usual $\bar{\partial}$ derivative of its matrix $\Phi(s)$ relative to the holomorphic frame s. Hence a bundle map Φ is holomorphic if and only if $\Phi_{\bar{z}} = 0$. Recall that Φ is connection-preserving if $\Phi_z = \Phi_{\bar{z}} = 0$, and combining this with (2.2), we have:

Proposition 2.1. *A bundle map Φ on a holomorphic Hermitian vector bundle E over a domain in $\mathbb{C}$ preserves the canonical connection if and only if both Φ and Φ^* are holomorphic.*

An isometric holomorphic bundle map Φ is connection-preserving since $\Phi^* = \Phi^{-1}$, which is necessarily holomorphic.

Given two Hermitian vector bundles E_1 and E_2 with connections D_1 and D_2; respectively, let Φ be a bundle map from E_1 to E_2. We say Φ is connection-preserving if

$$D_2 \Phi = \Phi D_1.$$

Fix local frames s_1 and s_2 for E_1 and E_2; respectively. Then Φ is connection-preserving if its matrix Φ relative to the two frame satisfies

$$d\Phi + \Theta_2(s_2)\Phi - \Phi\Theta_1(s_1) = 0, \tag{2.6}$$

where $\Theta_i(s_i)$ is the connection matrix of D_i with respect to the frame s_i.

For a bundle map between two Hermitian vector bundles, one can define its covariant derivative analogously, and Proposition 2.1 still holds (see [2] for details).

3. Geometric realization of $V^*(T)$

This section is devoted to establishing Theorem 1.3. Throughout this section, a connection means the canonical connection on a given holomorphic Hermitian vector bundle.

The following technical lemma (Proposition 1, [5]) is useful in this section.

Lemma 3.1. *Let Ω be a domain in $\mathbb{C}$ and $f(z,w)$ be a function on $\Omega \times \Omega$ which is holomorphic in z and anti-holomorphic in w. Then*

$$f(z,z) = 0$$

for all z in Ω if and only if f vanishes identically on $\Omega \times \Omega$.

Corollary 3.2. *Let S_1, S_2 be two operators commuting with T, then $\Gamma S_1 = (\Gamma S_2)^*$ if and only if $S_1 = S_2^*$.*

Proof. Sufficiency follows from the definition of Γ and it remains to show $\Gamma S_1 = (\Gamma S_2)^*$ implies $S_1 = S_2^*$. Take a holomorphic frame $\{\sigma_i(z)\}$ for $E(T)$ over an open subset Δ. Then by the spanning property $\bigvee_{\lambda \in \Delta} E(T)_\lambda = \mathcal{H}$, it suffices to show that

$$\big\langle S_1\sigma_i(z), \sigma_j(w)\big\rangle = \big\langle \sigma_i(z), S_2\sigma_j(w)\big\rangle$$

for all i, j and z, w in Δ. Since the frame is holomorphic, both sides of the identity above is holomorphic in z and anti-holomorphic in w, and $\Gamma S_1 = (\Gamma S_2)^*$ implies that the identity holds for $z = w$, so Lemma 3.1 can be applied and we are done. □

In general, a holomorphic Hermitian bundle does not admit a holomorphic orthonormal frame, but in the special case of holomorphic curves, there always exists a local holomorphic frame which is "normal" at one point (see Lemma 2.4 in [2]).

Lemma 3.3. *(See [2].) Given a holomorphic curve f over Ω and a point z_0 in Ω, there exists a holomorphic frame $\{\sigma_i(z)\}$ for E_f in a neighborhood Δ of z_0 such that $(\langle \sigma_i(z), \sigma_j(z_0)\rangle)$ is the identity matrix for all z in Δ.*

The local frame $\{\sigma_i\}$ given by Lemma 3.3 is called a normal frame. The matrix of a connection-preserving bundle map relative to a normal frame is very well behaved.

Proposition 3.4. *Let f be a holomorphic curve over a domain Ω in $\mathbb{C}$ and $\{\sigma_i\}$ be a normal frame over an open subset Δ at a point z_0. If Φ is a connection-preserving bundle map on E_f, then its matrix relative to $\{\sigma_i\}$ is a constant matrix which commutes with the metric matrix $(\langle \sigma_i(z), \sigma_j(w)\rangle)$ for all z, w in Δ.*

Proof. By Proposition 2.1, both Φ and Φ^* are holomorphic. If we denote by $\Phi(z)$ and $\Psi(z)$ the matrix of Φ and Φ^* relative to base $\{\sigma_i(z)\}$ of the fibre at z, then $\Phi(z)$ and $\Psi(z)$ are both holomorphic matrix-valued functions. If we set $h(z,w) = (\langle \sigma_i(z), \sigma_j(w)\rangle)$, then h is holomorphic in z and anti-holomorphic in w such that $h(z, z_0) = I$. By elementary linear algebra we have

$$\Psi(z) = h(z,z)\Phi^*(z)h^{-1}(z,z). \tag{3.1}$$

Combining this with Lemma 3.1, we have

$$\Psi(z) = h(z,w)\Phi^*(w)h^{-1}(z,w).$$

Let $w = z_0$, we see that

$$\Psi(z) = \Phi^*(z_0).$$

Thus $\Psi(z)$ is constant which we denote by Ψ. Our original identity becomes

$$\Psi = h(z,z)\Phi^*(z)h^{-1}(z,z),$$

taking adjoints we get

$$\Psi^* = h^{-1}(z,z)\Phi(z)h(z,z).$$

Another application of Lemma 3.1 yields

$$\Psi^* = h^{-1}(z,w)\Phi(z)h(z,w).$$

Taking $w = z_0$ again, we have

$$\Psi^* = \Phi(z).$$

Thus $\Phi(z)$ is constant (which we also denote by Φ) and $\Phi^* = \Psi$. By (3.1), both Φ and Ψ commute with $h(z,z)$, and thus commutes with $h(z,w)$ as well, in light of Lemma 3.1. □

Remark 3.5. From the proof of the above proposition, we see that if we fix a normal frame and a connection-preserving bundle map Φ, the matrix of Φ^* is just the adjoint of the matrix of Φ. Recall that a connection-preserving bundle map is necessarily holomorphic, and thus is determined by its restriction to any open subset Δ. Therefore the mapping defined by sending a connection-preserving bundle map to its matrix relative to a local normal frame is an injective $*$-homomorphism.

Now we complete the proof of Theorem 1.3.

Proof of Theorem 1.3. One direction is easy. For an operator S in $V^*(T)$, both S and S^* commutes with T and $(\Gamma(S))^* = \Gamma(S^*)$. Thus the condition of Proposition 2.1 is satisfied and $\Gamma(S)$ is connection-preserving.

We now establish the other direction: any connection preserving bundle map is induced by an operator in $V^*(T)$.

As before, we fix an open subset Δ and a local holomorphic frame $\{\sigma_i(z)\}$ for the holomorphic curve $E(T)$ normalized at a point z_0 in Δ.

By the previous proposition, the matrix of the connection-preserving bundle map relative to this frame is a constant matrix which we also denote by Φ such that

$$\Phi h(z,w) = h(z,w)\Phi,$$

where $h(z,w) = (\langle\sigma_i(z), \sigma_j(w)\rangle)$.

For any z in Δ, the bundle map defines a linear operator on the fibre $\ker(T-z)$ whose matrix relative to the base $\{\sigma_i(z)\}$ is Φ. Since eigenvectors belonging to different eigenvalues are linearly independent, these fibre maps together give a well-defined linear transform T_Φ on their algebraic linear span

$$\mathcal{H}_0 = \operatorname{span}_{z\in\Delta} \ker(T - z),$$

which is a dense subspace of $\mathcal{H}$.

For any z in Δ, $T_\Phi \ker(T - z) \subseteq \ker(T - z)$ by our construction, which implies that T_Φ commutes with T on $\ker(T - z)$, and thus on $\mathcal{H}_0$ as well.

We claim that T_Φ is *bounded*.

To this end, let us take an arbitrary vector f in $\mathcal{H}_0$. For such an f, there exist vectors $f_1, \dots, f_m$ with $f_i \in \ker(T - z_i)$ for some $z_1, \dots, z_m$ in Δ such that

$$f = f_1 + \cdots + f_m.$$

Since $\{\sigma_i\}$ is a frame, there exist mn complex numbers a_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, such that

$$f_i = \sum_{j=1}^{n} a_{ij}\sigma_j(z_i)$$

for any $1 \leqslant i \leqslant m$. To simplify notation, we write

$$\mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$$

and

$$\boldsymbol{\sigma}(z) = \big(\sigma_1(z), \sigma_2(z), \dots, \sigma_n(z)\big)^T.$$

Then $f_i = \mathbf{a}_i\boldsymbol{\sigma}(z_i)$ and $f = \mathbf{a}_1\boldsymbol{\sigma}(z_1) + \cdots + \mathbf{a}_m\boldsymbol{\sigma}(z_m)$.

Now

$$\begin{aligned}
\|f\|^2 &= \big\langle \mathbf{a}_1\boldsymbol{\sigma}(z_1) + \cdots + \mathbf{a}_m\boldsymbol{\sigma}(z_m), \mathbf{a}_1\boldsymbol{\sigma}(z_1) + \cdots + \mathbf{a}_m\boldsymbol{\sigma}(z_m)\big\rangle \\
&= \sum_{i,j=1}^{m} \mathbf{a}_i h(z_i, z_j)\mathbf{a}_j^* \\
&= (\mathbf{a}_1, \dots, \mathbf{a}_m)\big[h(z_i, z_j)\big](\mathbf{a}_1, \dots, \mathbf{a}_m)^*.
\end{aligned}$$

Here $(\mathbf{a}_1, \dots, \mathbf{a}_m)$ is a row of mn complex numbers and $[h(z_i, z_j)]$ is an $mn \times mn$ matrix whose $n \times n$ block at the (i, j) place is the matrix $h(z_i, z_j)$.

For example, if $m = 2$, there are only two points z_1 and z_2 involved and

$$\big[h(z_i, z_j)\big] = \begin{pmatrix} h(z_1, z_1) & h(z_1, z_2) \\ h(z_2, z_1) & h(z_2, z_2) \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}
\|T_\Phi f\|^2 &= \langle \mathbf{a}_1 \Phi \boldsymbol{\sigma}(z_1) + \cdots + \mathbf{a}_m \Phi \boldsymbol{\sigma}(z_m), \mathbf{a}_1 \Phi \boldsymbol{\sigma}(z_1) + \cdots + \mathbf{a}_m \Phi \boldsymbol{\sigma}(z_m) \rangle \\
&= \sum_{i,j=1}^{m} \mathbf{a}_i \Phi h(z_i, z_j) \Phi^* \mathbf{a}_j^* \\
&= (\mathbf{a}_1, \ldots, \mathbf{a}_m)(\Phi \otimes I_m)\big[h(z_i, z_j)\big]\big(\Phi^* \otimes I_m\big)(\mathbf{a}_1, \ldots, \mathbf{a}_m)^*
\end{aligned}$$

where $\Phi \otimes I_m$ is a block-diagonal matrix with Φ repeated m times on the diagonal.

Recall that $\Phi h(z_i, z_j) = h(z_i, z_j)\Phi$, which implies

$$(\Phi \otimes I_m)\big[h(z_i, z_j)\big] = \big[h(z_i, z_j)\big](\Phi \otimes I_m).$$

Note that $[h(z_i, z_j)]$ is a positive matrix, so we have

$$(\Phi \otimes I_m)\big[h(z_i, z_j)\big]^{\frac{1}{2}} = \big[h(z_i, z_j)\big]^{\frac{1}{2}}(\Phi \otimes I_m),$$

and

$$\big(\Phi^* \otimes I_m\big)\big[h(z_i, z_j)\big]^{\frac{1}{2}} = \big[h(z_i, z_j)\big]^{\frac{1}{2}}\big(\Phi^* \otimes I_m\big).$$

Consequently

$$(\Phi \otimes I_m)\big[h(z_i, z_j)\big]\big(\Phi^* \otimes I_m\big) = \big[h(z_i, z_j)\big]^{\frac{1}{2}}(\Phi \otimes I_m)\big(\Phi^* \otimes I_m\big)\big[h(z_i, z_j)\big]^{\frac{1}{2}},$$

thus

$$(\Phi \otimes I_m)\big[h(z_i, z_j)\big]\big(\Phi^* \otimes I_m\big) \leqslant \|\Phi \otimes I_m\|^2\big[h(z_i, z_j)\big] = \|\Phi\|^2\big[h(z_i, z_j)\big]$$

which implies that

$$\|T_\Phi f\| \leqslant \|\Phi\| \|f\|.$$

Here $\|\Phi\|$ is the standard matrix norm of Φ which dose not depend on f, hence the claim is proved. Since $\mathcal{H}_0$ is dense, T_Φ extends to a bounded operator on $\mathcal{H}$ and the extended operator still commutes with T. By our construction, $\Phi = \Gamma(T_\Phi)$ for the extended T_Φ.

We further claim that $(T_\Phi)^*$ commutes with T, which means T_Φ is in $V^*(T)$ and the proof of the theorem will be complete.

Let Ψ be the adjoint of the bundle map Φ, then as in the proof of Proposition 3.4, its matrix relative to the normal frame is also a constant matrix Ψ and $\Psi h(z, w) = h(z, w)\Psi$ for all z, w in Δ. Therefore, using the same argument, there exists a bounded operator T_Ψ commuting with T such that $\Psi = \Gamma(T_\Psi)$. By Corollary 3.2, $(T_\Phi)^* = T_\Psi$, hence T_Φ is in $V^*(T)$. □

Just as in Remark 1.2, we have:

Remark 3.6. For a Cowen–Douglas operator T, a local connection-preserving bundle map on $E(T)$ can be extended to a global connection-preserving bundle map induced by an operator in $V^*(T)$.

For a C^∞ vector bundle E on a planar domain with a given connection, bundle maps on E can be seen as sections of the tensor bundle $E \otimes E^*$ and a bundle map Φ is connection-preserving if and only if it is a parallel section of $E \otimes E^*$. Thus a connection-preserving bundle map is determined by its action on any fibre. In the case of holomorphic curves with canonical connection, this follows immediately from Proposition 3.4.

Consequently, for a Cowen–Douglas operator T in $\mathcal{B}_n(\Omega)$ and any point w_0 in Ω, an operator in $V^*(T)$ is determined by its action on $\ker(T - w_0)$. In particular, we have:

Corollary 3.7. *For a Cowen–Douglas operator T, $V^*(T)$ is finite dimensional.*

We end this section with a straightforward proof of this corollary in operator theory, which is of independent interest.

Proof. Without loss of generality, we assume $w = 0$. Since $\ker T$ is finite dimensional, it suffices to show that if S is an operator in $V^*(T)$ such that $S|_{\ker T} = 0$, then $S = 0$.

In fact, since $\mathcal{H} = \ker T \oplus \operatorname{ran} T^*$ (note that $\operatorname{ran} T^*$ is closed in this case), we have

$$S\mathcal{H} = ST^*\mathcal{H} = T^*S\mathcal{H} = T^*ST^*\mathcal{H} = \left(T^*\right)^2 S\mathcal{H} = \cdots \subseteq \bigcap_{k=1}^{\infty} \operatorname{ran}\left(T^*\right)^k.$$

Note that the spanning property implies that $\bigvee_{k=1}^{\infty} \ker T^k = \mathcal{H}$ (Section 1.7, [2]), hence $\bigcap_{k=1}^{\infty} \operatorname{ran}(T^*)^k = 0$, as desired. □

4. Connection-preserving bundle maps on $E(T)$

In this section, we study connection-preserving bundle maps on $E(T)$ and provide a characterization of $V^*(T)$ in terms of geometric invariants. Before proceeding, we would like to say more about reducing subbundles of holomorphic Hermitian vector bundles.

Let E be a holomorphic Hermitian vector bundle with canonical connection D. Given a reducing subbundle E' of E, we can chose holomorphic frames s' and s'' of E' and $E'^{\perp}$; respectively such that $s = \{s', s''\}$ forms a holomorphic frame for E. Relative to this frame, the metric matrix of E decomposes into two blocks. Therefore by (2.4), (2.5) the matrices of the canonical connection D and curvature $\mathcal{K}$ also decompose into two blocks. By the following representation of covariant derivatives:

$$\mathcal{K}_z(s) = \partial\mathcal{K}(s) + \left[\partial h h^{-1}, \mathcal{K}(s)\right],$$
$$\mathcal{K}_{\bar{z}}(s) = \bar{\partial}\mathcal{K}(s),$$

we see that the matrices of the covariant derivatives of the curvature to all orders also decompose into two blocks relative to this frame. In particular, reducing subbundles are D-invariant. The following result (see Proposition 4.18, Chapter 1 in [11]) asserts that the converse is also true, which can be used to identify reducing subbundles of holomorphic Hermitian vector bundles. We include the proof for the convenience of the readers.

Proposition 4.1. *(See [11].) Let E be a holomorphic Hermitian vector bundle and D the canonical connection. Let E' be a C^∞ subbundle and E'' be the orthogonal complement of E' in E.*

If E' is invariant under D, both E' and E'' are D-invariant holomorphic subbundles of E and they give a holomorphic orthogonal decomposition:

$$E = E' \oplus E''.$$

Proof. As is well known, the canonical connection D can be decomposed as

$$D = D' + \bar{\partial}$$

with

$$D' : \mathcal{E}^0(\Omega, E) \to \mathcal{E}^{1,0}(\Omega, E),$$
$$\bar{\partial} : \mathcal{E}^0(\Omega, E) \to \mathcal{E}^{0,1}(\Omega, E).$$

Since E' is invariant under D, so is E'' because D is metric preserving. Let s be a holomorphic section of E and $s = s' + s''$ be its C^∞ decomposition with respect to $E = E' \oplus E''$. It suffices to show s' and s'' are holomorphic sections. Since $D = D' + \bar{\partial}$ and s is holomorphic, we have $Ds = D's$. On the other hand, $Ds = Ds' + Ds''$ and $D's = D's' + D's''$, which implies $Ds' = D's'$ and $Ds'' = D's''$. Therefore $\bar{\partial}s' = 0$ and $\bar{\partial}s'' = 0$, as desired. □

Since the canonical connection on a holomorphic Hermitian vector bundle is unique, the canonical connection on a reducing subbundle is just the restriction of the original one.

As stated in the introduction, our investigation is based on a representation theorem of Cowen and the second author, called the C^∞ block diagonalization of connections. We begin with some necessary terminologies before introducing this result.

Let E be a C^∞ Hermitian vector bundle of rank n over a domain Ω in $\mathbb{C}$ with metric-preserving connection D and curvature $\mathcal{K}$. We denote by $\mathscr{A}$ the algebra of bundle maps generated by the curvature $\mathcal{K}$ and its covariant derivatives $\mathcal{K}_{z^i\bar{z}^j}$ to all orders. Since $\mathcal{K}$ is self-adjoint and the identity (2.2) holds, $\mathscr{A}$ is self-adjoint.

Let s be a C^∞ orthonormal frame of E over an open subset Δ of Ω. For a bundle map Φ on E and z in Δ, let $\Phi(z)$ be the induced fibre map on the fibre E_z and $\Phi(s)(z)$ be the matrix of $\Phi(z)$ relative to the base $s(z)$. We denote by $\mathscr{A}(z)$ the set of linear transforms on the fibre E_z induced by bundle maps in $\mathscr{A}$ and $\mathscr{A}(s)(z)$ the matrix algebra generated by the matrices $\Phi(s)(z)$ for Φ in $\mathscr{A}$, then $\mathscr{A}(s)(z)$ is a self-adjoint matrix algebra in $M_n(\mathbb{C})$ since s is orthonormal.

It is well known that any self-adjoint matrix algebra is the direct sum of full matrix algebras with multiplicity. More precisely, for any self-adjoint matrix algebra, there exist two tuples of positive integers $\mathscr{M} = (m_1, \ldots, m_r)$ and $\mathscr{N} = (n_1, \ldots, n_r)$, such that the algebra consists of matrices of the form $A_1 \otimes I_{m_1} \oplus \cdots \oplus A_r \otimes I_{m_r}$, where A_i is an $n_i \times n_i$ matrix repeated m_i times on the diagonal, we denote such an algebra by $M(\mathscr{N}, \otimes\mathscr{M})$.

For example, $M((n_1, n_2), \otimes(2, 1))$ is the algebra of matrices of the form

$$A_1 \otimes I_2 \oplus A_2 \otimes I_1 = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix},$$

where A_1 is an $n_1 \times n_1$ matrix and A_2 is an $n_2 \times n_2$ matrix.

Now we can state the theorem on block diagonalization of connections (see Proposition 2.5 in [3], also see Lemma 3.2 in [2] for a special case).

Theorem 4.2. *(See [3].) Let E be a C^∞ Hermitian vector bundle of rank n over an open subset Ω in $\mathbb{C}$, with metric-preserving connection D. For any point z_0 off a non-where dense subset of Ω, there exist two tuples of integers $\mathscr{M} = (m_1, \ldots, m_r)$, $\mathscr{N} = (n_1, \ldots, n_r)$, a neighborhood Ω_0 of z_0 and a C^∞ orthonormal frame s for E over Ω_0 with the properties*:

$$\mathscr{A}(s)(z) = M(\mathscr{N}, \otimes\mathscr{M})$$

for all z in Ω_0, where $\mathscr{A}$ is the algebra of bundle maps generated by the curvature $\mathcal{K}$, and its covariant derivatives $\mathcal{K}_{z^i\bar{z}^j}$ to all orders. Moreover,

$$\Theta(s) = \Theta_1 \otimes I_{m_1} \oplus \cdots \oplus \Theta_r \otimes I_{m_r},$$

where $\Theta(s)$ is the matrix of connection 1-forms of D relative to the frame s and Θ_i are C^∞ $n_i \times n_i$ matrices with 1-form entries defined on Ω_0.

Remark 4.3. There are various ways to understand this theorem.

(i) The algebra $\mathscr{A}(s)(z)$ does not depend on the point z in Ω_0.
(ii) The theorem asserts that the connection matrix has a block diagonal form, thus each block corresponds to a subbundle invariant under D. Explicitly, for any $1 \leqslant i \leqslant r$, the block $\Theta_i \otimes I_{m_i}$ corresponds to m_i D-invariant subbundles of rank n_i. We denote these subbundles by $E_{i1}, \ldots, E_{im_i}$. With respect to this decomposition, the frame s can be written as $s = \{s_{ij}\}$ where s_{ij} is an orthonormal frame for E_{ij}.
(iii) By definitions, the curvature as well as its partial derivatives are determined by the connections, while the theorem implies that the connection can be determined by the curvature in some sense.

If E is a holomorphic Hermitian vector bundle with canonical connection D, then D-invariant subbundles are actually reducing subbundles for E by Proposition 4.1. Therefore we can apply Theorem 4.2 to obtain a collection of mutually orthogonal reducing subbundles $\{E_{ij}\}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant m_i$ with rank $E_{ij} = n_i$, such that

$$E = E_{11} \oplus \cdots \oplus E_{1m_1} \oplus \cdots \oplus E_{r1} \oplus \cdots \oplus E_{rm_r}.$$

If we apply the theorem to the bundle $E(T)$ with canonical connection for a Cowen–Douglas operator T, then $\{E_{ij}\}$ correspond to reducing subspaces $\{\mathcal{H}_{ij}\}$ such that

$$\mathcal{H} = \mathcal{H}_{11} \oplus \cdots \oplus \mathcal{H}_{1m_1} \oplus \cdots \oplus \mathcal{H}_{r1} \oplus \cdots \oplus \mathcal{H}_{rm_r}. \tag{4.1}$$

We will show that $\mathcal{H}_{ij}$ are minimal and (4.1) gives a canonical decomposition of $\mathcal{H}$ into minimal reducing subspaces. To get a full understanding of that, we first recall some elementary facts on von Neumann algebras. In light of Corollary 3.7, we concentrate on the finite dimensional case.

Given a von Neumann algebra M, we denote its center by $Z(M)$ and its identity by 1_M. Two projections p and q in M are said to be *equivalent* if there exists an element u in M such that $u^*u = p$, $uu^* = q$. A projection p in M is said to be minimal if for any projection q in M, $q \leqslant p$ implies $q = 0$ or $q = p$.

If M is a finite dimensional von Neumann algebra, there exists finitely many mutually orthogonal minimal projections $q_1, \dots, q_k$ in M such that

$$1_M = q_1 + \cdots + q_k. \tag{4.2}$$

The center $Z(M)$ is also finite dimensional, thus there are finitely many mutually orthogonal minimal central projections (i.e. minimal projections in $Z(M)$) $p_1, \dots, p_r$, such that

$$1_M = p_1 + \cdots + p_r.$$

One can show, as a routine exercise, that (i) for any minimal projection q in M, there exist exactly one index i such that $qp_i = q$ (equivalently, $q \leqslant p_i$) and $qp_j = 0$ for $j \neq i$, (ii) two minimal projections in M are equivalent if and only if they are dominated by the same minimal central projection.

By (i), we can rearrange the minimal projections in (4.2) such that

$$1_M = q_{11} + \cdots + q_{1m_1} + \cdots + q_{r1} + \cdots + q_{rm_r} \tag{4.3}$$

with $q_{i1} + \cdots + q_{im_i} = p_i$, and by (ii), q_{ij} and $q_{i'j'}$ are equivalent if and only if $i = i'$. We call (4.3) a canonical decomposition.

Now we go back to the Cowen–Douglas operator T. Reducing subspaces of T can be identified with projections in $V^*(T)$ and it is easy to check that two reducing subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are unitarily equivalent if and only if the their corresponding projections in $V^*(T)$ are equivalent. The following theorem says (4.1) is a canonical decomposition in the sense we discussed above, while the proof is geometric.

Theorem 4.4. *Let T be a Cowen–Douglas operator and $E(T)$ be its associated holomorphic Hermitian vector bundle with reducing subbundles $\{E_{ij}\}$ given by the block diagonalization of the canonical connection. Let $\{\mathcal{H}_{ij}\}$ be the corresponding reducing subspaces. Then*:

(i) *The reducing subspaces $\{\mathcal{H}_{ij}\}$ are minimal.*
(ii) *$\mathcal{H}_{ij}$ and $\mathcal{H}_{i'j'}$ are unitarily equivalent if and only if $i = i'$.*

Proof. (i) It suffices to show the bundle E_{ij} is irreducible. Suppose conversely that

$$E_{ij} = F_1 \oplus F_2$$

for two orthogonal holomorphic subbundles F_1 and F_2, then E_{ij} admits a holomorphic frame $\widetilde{s} = \{\widetilde{s}_1, \widetilde{s}_2\}$, where $\widetilde{s}_i$ is a holomorphic frame for F_i. Let $\mathscr{A}(E_{ij})$ be the restriction of $\mathscr{A}$ on E_{ij}, then as mentioned in the beginning of the section, for z in Ω_0, the matrix of any linear map in $\mathscr{A}(E_{ij})(z)$ should take a block diagonal form relative to the base $\widetilde{s}(z) = \{\widetilde{s}_1(z), \widetilde{s}_2(z)\}$ for the fibre E_{ij_z}. While on the other hand, $\mathscr{A}(E_{ij})(z)$ contains all linear transformations on the fibre

E_{ij_z} since by Theorem 4.2, $\mathscr{A}(E_{ij})(s_{ij})(z)$ is the full matrix algebra $M_{n_i}(\mathbb{C})$ relative to the frame s_{ij} mentioned in Remark 4.3, a contradiction.

(ii) Without loss of generality, we prove the statement for $r = 2$, $m_1 = 2$, $m_2 = 1$. In this case, relative to the frame s in Theorem 4.2, we have a decomposition

$$E(T) = E_{11} \oplus E_{12} \oplus E_{21}.$$

Here $s = \{s_{11}, s_{12}, s_{21}\}$ where s_{11}, s_{12}, s_{21} are orthonormal frames for E_{11}, E_{12}, E_{21} respectively as in Remark 4.3. The matrix algebra $\mathscr{A}(s)(z)$ contains all matrices of the form

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix}$$

and the connection matrix is of the form

$$\begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_1 & 0 \\ 0 & 0 & \Theta_2 \end{pmatrix}.$$

In light of Theorem 1.1, it suffices to show that E_{11} and E_{12} are equivalent while E_{12} and E_{21} are not equivalent.

That E_{11} and E_{12} are equivalent is straightforward. We define an isometric bundle map by sending the orthonormal frame s_{11} to s_{12}. We claim that this bundle map is holomorphic, and hence implements an equivalence of the two bundles.

In fact, by Proposition 2.1, it suffices to show this bundle map is connection preserving. Since the matrix of this bundle map relative to the frames s_{11} and s_{12} is the constant identity matrix and the connection matrices relative to the two frames are the same, we see that (2.6) holds. Hence the claim follows.

Next we show that there exists no isometric connection-preserving bundle map from E_{12} to E_{21}. If there exists such a bundle map Φ, then E_{12} and E_{21} are of the same rank and by the discussions in Section 2, Φ preserves the curvatures as well as their covariant derivatives to all orders. Hence Φ commutes with the restriction of $\mathscr{A}$ to E_{12} and E_{21}. Suppose $\operatorname{rank} E_{12} = \operatorname{rank} E_{21} = k$, then by Theorem 4.2, for any fixed z in Ω_0 and any two $k \times k$ matrices A_1 and A_2, there exists a bundle map in $\mathscr{A}$ such that the matrices of its restriction to $E_{12}(z)$ and $E_{21}(z)$ relative to the base $s_{11}(z)$ and $s_{12}(z)$ are A_1 and A_2 respectively. So if $\Phi(z)$ is the matrix of Φ relative to the bases $s_{11}(z)$ and $s_{12}(z)$, then

$$\Phi(z)A_1 = A_2\Phi(z),$$

which forces $\Phi(z)$ to be zero since A_1 and A_2 can be arbitrarily chosen. $\square$

We give the promised geometrical characterization of $V^*(T)$.

Theorem 4.5. *For a Cowen–Douglas operator T in $\mathfrak{B}_n(\Omega)$, the von Neumann algebra $V^*(T)$ is isomorphic to the commutant of the matrix algebra $M(\mathscr{N}, \otimes\mathscr{M})$ in $M_n(\mathbb{C})$, where $M(\mathscr{N}, \otimes\mathscr{M})$ is given by the block diagonalization of the canonical connection on $E(T)$.*

Proof. By Theorem 1.3, it suffices to identify connection-preserving bundle maps on $E(T)$ with the commutant of $M(\mathscr{N}, \otimes\mathscr{M})$.

Denote the algebra of connection-preserving bundle maps on $E(T)$ by $\mathcal{V}$. We claim that for any bundle map Φ in $\mathcal{V}$, the matrix of Φ relative to the orthonormal frame s given in Theorem 4.2 is a constant matrix and lies in $M'(\mathscr{N}, \otimes\mathscr{M})$.

In fact, for a fixed z in Ω_0, let $\Phi(z)$ be the matrix of Φ relative to the base $s(z)$, then since Φ is connection-preserving, it commutes with every bundle map in $\mathscr{A}$, thus by Theorem 4.2, $\Phi(z)$ commutes with every matrix in $M(\mathscr{N}, \otimes\mathscr{M})$.

Moreover, $\Phi(z)$ commutes with the connection matrix $\Theta(z)$ since $\Theta(z)$ is just a matrix in $M(\mathscr{N}, \otimes\mathscr{M})$ tensored with a 1-form, so

$$\big[\Theta(z), \Phi(z)\big] = 0.$$

Recall that the matrix of a connection-preserving bundle map satisfies

$$d\Phi(z) + \big[\Theta(z), \Phi(z)\big] = 0,$$

therefore $d\Phi(z) = 0$, and $\Phi(z)$ is constant.

Now we have a map Λ from $\mathcal{V}$ to $M'(\mathscr{N}, \otimes\mathscr{M})$ sending a connection-preserving bundle map to its matrix relative to the frame s, which is well defined. Note that since the frame is orthonormal, Λ is a $*$-homomorphism. Moreover, Λ is injective since a connection-preserving bundle map is determined by its action on any open subset. Λ is surjective since any constant matrix in $M'(\mathscr{N}, \otimes\mathscr{M})$ satisfies (2.3). Thus a local bundle map given by such a matrix relative to the frame s is connection-preserving on Ω_0. By Remark 3.6, this local bundle map can be extended to a connection-preserving bundle map on all Ω, completing the proof. □

The commutant of $M(\mathscr{N}, \otimes\mathscr{M})$ consists of matrices of the form

$$I_{n_1} \otimes B_1 \oplus \cdots \oplus I_{n_r} \otimes B_r,$$

where B_i is an $m_i \times m_i$ matrix. We see that $M'(\mathscr{N}, \otimes\mathscr{M})$ is abelian if and only if $m_i = 1$ for all i. Thus we have the following

Corollary 4.6. *The von Neumann algebra $V^*(T)$ is abelian if and only if there is no multiplicity in the block diagonalization of the canonical connection on $E(T)$.*

In general, it is not easy to compute the matrix algebra $M(\mathscr{N}, \otimes\mathscr{M})$ explicitly for an arbitrary Cowen–Douglas operator. We discuss a special kind of operator T_E, called the bundle shift. The adjoint of T_E lies in the Cowen–Douglas class and $V^*(T_E)$ can be identified via the topological construction of a certain flat unitary bundle E.

The bundle shift T_E was introduced in [1] and we give a quick review of its definition. Let Ω be a bounded domain in $\mathbb{C}$ whose boundary consists of finitely many analytic Jordan curves. A flat unitary bundle over Ω is a holomorphic Hermitian vector bundle which locally admits orthonormal holomorphic frames (or equivalently, the transition functions are constant unitary matrices). It is well known that any flat unitary bundle over Ω is equivalent to a canonical flat bundle arising from a unitary representation of the fundamental group $\pi_1(\Omega)$. Let us briefly recall this construction.

By the uniformization theorem, there is a holomorphic covering map $\pi : \mathbb{D} \to \Omega$ where $\mathbb{D}$ is the unit disc. Let $\mathscr{U}(n)$ be the group of unitary operators on $\mathbb{C}^n$ and a unitary representation of $\pi_1(\Omega)$ is a homomorphism $\alpha : \pi_1(\Omega) \to \mathscr{U}(n)$. Define an action of $\pi_1(\Omega)$ on $\mathbb{D} \times \mathbb{C}^n$ by

$$A : (z, \xi) \mapsto \big(Az, \alpha(A)\xi\big)$$

for $A \in \pi_1(\Omega)$, $z \in \mathbb{D}$, and $\xi \in \mathbb{C}^n$. (We identify $\pi_1(\Omega)$ with the covering transformation group acting on $\mathbb{D}$.) Then the quotient space $\mathbb{D} \times \mathbb{C}^n/\pi_1(\Omega)$ of this action with the obvious projection onto Ω gives a flat unitary bundle of rank n over Ω.

Given a flat unitary bundle E over Ω, one can construct a Hilbert space H_E^2 consisting of holomorphic sections f of E such that $\|f(z)\|_{E_z}^2$ has a harmonic majorant. The bundle shift T_E is defined on H_E^2 by $T_E(f) = zf$. One can show that T_E^* lies in $\mathcal{B}_n(\Omega^*)$, where Ω^* is the complex conjugate of Ω (Theorem 3, [1]).

A fundamental result on the bundle shift is the following (Theorem 6, [1]).

Theorem 4.7. *(See [1].) If E and F are flat unitary bundles over Ω, then the bundle shifts T_E and T_F are unitarily equivalent if and only if E and F are equivalent.*

Remark 4.8. Any two flat unitary bundles of the same rank are locally equivalent since they admit local orthonormal holomorphic frames, while the theorem requires that the isometric holomorphic bundle map can be defined globally.

Moreover, we have a characterization of the von Neumann algebra $V^*(T_E)$ (Theorem 7, [1]):

Theorem 4.9. *(See [1].) For a rank n flat unitary bundle E over Ω arising from a unitary representation α of $\pi_1(\Omega)$, the von Neumann algebra $V^*(T_E)$ is isomorphic to the commutant of $C^*(\alpha)$ in $M_n(\mathbb{C})$, where $C^*(\alpha)$ is the C^* algebra generated by the range of α.*

A geometric interpretation of Theorem 4.9 in terms of bundle maps, which is related to our investigation, is the following:

Corollary 4.10. *For a rank n flat unitary bundle E over Ω arising from a unitary representation α of $\pi_1(\Omega)$, any operator in $V^*(T_E)$ is induced by a (global)connection-preserving bundle map on E.*

Proof. For one thing, the connection matrix $\Theta(s)$ is zero for any local orthonormal holomorphic frame s by (2.4). Thus for a fixed matrix Φ in $M_n(\mathbb{C})$, a local bundle map defined by this constant matrix relative to the frame s satisfies (2.3).

On the other hand, one can check that the transition matrix of two different local orthonormal holomorphic frames whenever their defining domains overlap is nothing but $\alpha(A)$ for some $A \in \pi_1(\Omega)$ (in fact, the local holomorphic orthonormal frames arise from branches of local inverses of the covering map), therefore when Θ lies in the commutant of $C^*(\alpha)$, $\Phi = \alpha^{-1}(A)\Phi\alpha(A)$, which is exactly the condition assuring that the locally defined connection-preserving bundle maps glue to a global one. That such a bundle map induces an operator in $V^*(T_E)$ follows by tracing back the original proof of Theorem 4.9 and is omitted here. □

The following consequence of Theorem 4.9 can be seen as a complement of our main results.

Corollary 4.11. *For any self-adjoint subalgebra $\mathcal{A}$ of $M_n(\mathbb{C})$, there exists a Cowen–Douglas operator T such that $V^*(T) \simeq \mathcal{A}$.*

Proof. It follows from general theory of self-adjoint matrix algebras that the commutant algebra $\mathcal{A}'$ of $\mathcal{A}$ can be generated by finitely many, say, k unitary matrices. Take a planar domain Ω with k holes so that $\pi_1(\Omega)$ is a free group of k generators. The map α defined by taking each generator of $\pi_1(\Omega)$ to one of the unitary matrices generating $\mathcal{A}'$ extends to a unitary representation α of $\pi_1(\Omega)$ with $C^*(\alpha) = \mathcal{A}'$. By Theorem 4.9, $V^*(T_E) \simeq \mathcal{A}'' = \mathcal{A}$, where E is the flat unitary bundle arising from α. □

Appendix A

To better understand the block diagonalization theorem, we describe an alternative proof of the sufficiency part of Theorem 1.3, which is based on the discussions in Section 4.

The idea is to replace the normal frame given by Lemma 3.3 by the orthonormal frame given in Theorem 4.2. If we can verify Proposition 3.4 and Remark 3.5 for this orthonormal frame, then all the arguments in the proof of Theorem 1.3 remain valid and we have the same conclusion.

By the proof of Theorem 4.5, the matrix Φ of the connection-preserving bundle map relative to the orthonormal frame s in Theorem 4.2 is constant and lies in the commutant of $M(\mathscr{N}, \otimes\mathscr{M})$. We write $s = \{s_1, \ldots, s_n\}$ for n C^∞ sections $s_1, \ldots, s_n$. To verify Proposition 3.4, we only need to check that for any z, w in Δ, the matrix $(\langle s_i(z), s_j(w)\rangle)$ lies in $M(\mathscr{N}, \otimes\mathscr{M})$. Note that we cannot apply Lemma 3.1 for the non-holomorphic frame s.

Without loss of generality, we assume $r = 2$, $m_1 = 2$, $m_2 = 1$ as in the proof of Theorem 4.4 so that the bundle $E(T)$ has the decomposition

$$E(T) = E_{11} \oplus E_{12} \oplus E_{21}.$$

We need to show that the matrix $(\langle s_i(z), s_j(w)\rangle)$ is of the form

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix}.$$

Write $\{s_i\} = \{\mu_i\} \cup \{\eta_i\} \cup \{\nu_i\}$ where $\{\mu_i\}$, $\{\eta_i\}$ and $\{\nu_i\}$ are orthonormal frames for E_{11}, E_{12} and E_{21} respectively.

Take arbitrary sections f_1, f_1 and f_3 of E_{11}, E_{12} and E_{21}; respectively. Since $f_1(z) \in \mathcal{H}_{11}$, $f_2(w) \in \mathcal{H}_{12}$ and $\mathcal{H}_{11}$ and $\mathcal{H}_{12}$ are mutually orthogonal reducing subspaces, $\langle f_1(z), f_2(w)\rangle = 0$. Similarly, we have $\langle f_1(z), f_3(w)\rangle = 0$ and $\langle f_2(z), f_3(w)\rangle = 0$. This implies that $(\langle s_i(z), s_j(w)\rangle)$ is of the form

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}.$$

We claim that $A_1 = A_2$, which gives the desired form. In fact, by Theorem 4.4, the bundle map defined by sending $\{\mu_i\}$ to $\{\eta_i\}$ is induced by a unitary operator from $\mathcal{H}_{11}$ to $\mathcal{H}_{12}$, and thus

$$\big(\langle \mu_i(z), \mu_j(w)\rangle\big) = \big(\langle \eta_i(z), \eta_j(w)\rangle\big)$$

as desired.

Since the frame is orthonormal and the commutant of $M(\mathscr{N}, \otimes\mathscr{M})$ is a self-adjoint algebra, Remark 3.5 is trivial for this frame.

References

[1] M.B. Abrahamse, R.G. Douglas, A class of subnormal operators related to multiply-connected domains, Adv. Math. 19 (1976) 106–148.
[2] M.J. Cowen, R.G. Douglas, Complex geometry and operator theory, Acta Math. 141 (1978) 187–261.
[3] M.J. Cowen, R.G. Douglas, Equivalence of connections, Adv. Math. 56 (1985) 39–61.
[4] R.G. Douglas, S. Sun, D. Zheng, Multiplication operators on the Bergman space via analytic continuation, arXiv:0901.3787v1.
[5] M. Englis, Density of algebras generated by Toeplitz operator on Bergman spaces, Ark. Mat. 30 (1992) 227–243.
[6] K. Guo, Operator theory and von Neumann algebras, preprint.
[7] K. Guo, H. Huang, On multiplication operators of the Bergman space: Similarity, unitary equivalence and reducing subspaces, J. Operator Theory, in press.
[8] K. Guo, H. Huang, Multiplication operators defined by covering maps on the Bergman space: The connection between operator theory and von Neumann algebras, J. Funct. Anal. 260 (4) (2011) 1219–1255.
[9] K. Guo, S. Sun, D. Zheng, C. Zhong, Multiplication operators on the Bergman space via the Hardy space of the bidisk, J. Reine Angew. Math. 628 (2009) 129–168.
[10] J. Hu, S. Sun, X. Xu, D. Yu, Reducing subspace of analytic Toeplitz operators on the Bergman space, Integral Equations Operator Theory 49 (2004) 387–395.
[11] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Princeton University Press and Iwanami Shoten, 1987.
[12] C. Jiang, K. Ji, Similarity classification of holomorphic curves, Adv. Math. 215 (2007) 446–468.
[13] R. Wells, Differential Analysis on Complex Manifolds, Springer, New York, 1973.
[14] K. Zhu, Reducing subspaces for a class of multiplication operators, J. Lond. Math. Soc. 62 (2000) 553–568.